

# Uncertainty Transformation via Hopf Bifurcation in Fast-Slow Systems

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## Abstract

Propagation of uncertainty in dynamical systems is a significant challenge. Here we focus on random multiscale ordinary differential equation models. In particular, we study Hopf bifurcation in the fast subsystem for random initial conditions. We show that a random initial condition distribution can be transformed during the passage near a delayed/dynamic Hopf bifurcation: (I) to certain classes of symmetric copies, (II) to an almost deterministic output, (III) to a mixture distribution with differing moments, and (IV) to a very restricted class of general distributions. We prove under which conditions the cases (I)-(IV) occur in certain classes vector fields.

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**Keywords:** Fast-slow systems, Hopf bifurcation, random initial condition, uncertainty propagation.

## 1 Introduction

Many mathematical models of complex systems contain an inherent element of uncertainty. From one perspective, it is a strength of theoretical models to abstract, simplify, and reduce a real system into a conceptual form. Modelling the neglected, unknown, or different-scale processes can often be accomplished using probabilistic models. The challenge is then to quantify uncertainty, i.e., to explain what effect random terms have in comparison to the purely deterministic system.

Here we study the scenario when we do not have exact information about the initial condition. Suppose we model the initial condition as a random variable with a given distribution. Then the question is how the probability distribution is propagated by the dynamics? If the dynamical system contains an instability, e.g., a saddle-like structure in phase space, it is possible that a small random error in the initial condition can lead to widely different outcomes in the dynamics; see Figure 1(a). If all initial conditions in the distribution are

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attracted to a single stable attractor, then the randomness could probably have been omitted in the first place; see Figure 1(b). The critical cases are systems, which display transient and/or unstable behavior for a certain limited period of time in phase space; see Figure 1(c). This is the case considered in this paper.

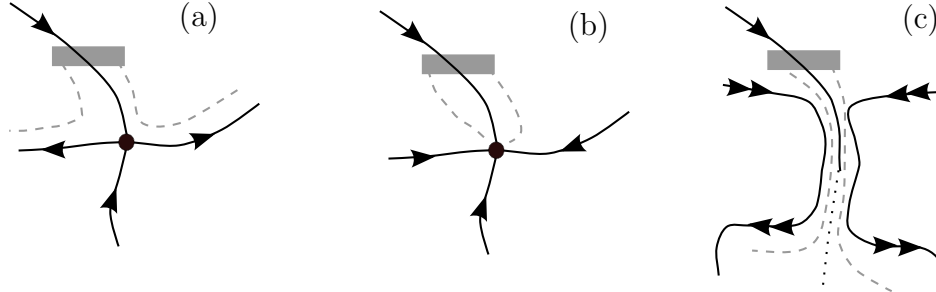


Figure 1: Sketch of several situations including uncertain initial conditions (grey rectangles). The flow is indicated with arrows, steady states are marked as circles and propagation of certain points in the set of potential initial conditions is given by dashed grey curves. (a) Saddle steady state and exponentially diverging random initial conditions. (b) Stable node and exponentially contracted random initial conditions. (c) Transient instability.

A passage near many instabilities is frequently modeled using multiple time scale dynamical systems. A fundamental subclass are fast-slow ordinary differential equations (ODEs). Slowly-drifting variables may bring the system towards an instability of certain fast variables. Near the instability an intricate interplay between the different classes of variables emerges. Detailed studies of many class of fast-slow bifurcation scenarios exist, see e.g. [8, 15, 22] and the extensive references in [16]. The topic is sometimes referred to as 'dynamic bifurcation' or 'delayed bifurcation' and has a long history [11, 23, 27, 19, 3, 10]; for the delayed Hopf case considered in this paper see [1, 2, 12, 20, 21, 25].

On the side of stochastic fast-slow systems the case of additive or multiplicative stochastic terms has been studied for multiscale stochastic ordinary differential equations (SODEs) from different perspectives [4, 13] including bifurcation delay [17, 18, 7, 29]; see [16, Sec.15.10] for more detailed references. The case of multiscale random ordinary differential equations (RODEs) has been explored a lot less up to now. Regarding delayed bifurcations of RODEs, a particular model case arising in pattern formation [5, 28] seems to be the first study.

Here we concentrate on the abstract theory of delayed Hopf bifurcation for RODEs. The Hopf case is definitely among the most interesting cases for bifurcation delay (see the review of results in Section 2 and the references mentioned above). The results of this work relate an input distribution  $\mu_0$  of initial conditions to an output distribution  $\mu_*$ . We briefly state the conclusions in non-technical terms:

- (I) The initial distribution  $\mu_0$  can be transformed via only a restricted set within the class of orthogonal transformations; certain reflections are allowed (Theorem 3.1) while general translations cannot occur (Theorem 3.2).
- (II) Given  $\mu_0$  there exists a vector field mapping it to a prescribed delta-distribution in a singular limit (Theorem 3.4) and the delta-distribution deforms to an approximate identity (Corollary 3.6).

- (III) For large classes of given real-analytic vector fields we obtain mixture measures for  $\mu_*$  (Theorem 3.7); the moments of  $\mu_*$  are computable in many cases (Proposition 3.9) and (Proposition 3.10).
- (IV) For general given  $\mu_0$  and  $\mu_*$ , there is no real-analytic vector field which maps  $\mu_0$  to  $\mu_*$  under delayed Hopf bifurcation (Theorem 3.11 and Proposition 3.15).

In summary, we have shown that the problem of propagating uncertainty through regions with bifurcations displays interesting behaviour, even for the simple case of random initial condition and the codimension-one Hopf bifurcation. Although there are many computational studies and approaches via inverse problems to uncertainty quantification, the route via instabilities and multiscale bifurcation normal form theory seems to be a wide-open direction for future work.

## 2 Deterministic Delayed Hopf Bifurcation

We review basic results about deterministic delayed Hopf bifurcation to fix the notation and the setup. Consider a compact interval  $\mathcal{I} := [0, \varepsilon_*]$  for some sufficiently small  $\varepsilon_* > 0$ . Let  $\varepsilon > 0$ ,  $\varepsilon \in \mathcal{I}$ , be a parameter representing the time scale separation. Consider the three-dimensional *fast-slow system*

$$\begin{aligned} \varepsilon \frac{dx_1}{d\tau} &= \varepsilon \dot{x}_1 = f_1(x_1, x_2, y, \varepsilon), \\ \varepsilon \frac{dx_2}{d\tau} &= \varepsilon \dot{x}_2 = f_2(x_1, x_2, y, \varepsilon), \\ \frac{dy}{d\tau} &= \dot{y} = g(x_1, x_2, y, \varepsilon), \end{aligned} \tag{1}$$

where  $f = (f_1, f_2)^\top : \mathbb{R}^3 \times \mathcal{I} \rightarrow \mathbb{R}^2$ ,  $g : \mathbb{R}^3 \times \mathcal{I} \rightarrow \mathbb{R}$  are maps in a suitable function space  $\mathcal{X}$ ,  $x = (x_1, x_2)^\top \in \mathbb{R}^2$  are the *fast variables* and  $y \in \mathbb{R}^1$  is the *slow variable*. We also refer to (1) as a  $(2, 1)$ -fast-slow system due the dimensions of the sets of variables. We restrict the analysis to suitable subsets of phase space with  $x \in \mathcal{K}_x \subset \mathbb{R}^2$ ,  $y \in \mathcal{K}_y \subset \mathbb{R}$ , where  $\mathcal{K}_x$  will always be compact. We are going to need  $\mathcal{X} = C^k$  for some  $k \in \mathbb{N}$  with  $k \geq 3$ , or  $\mathcal{X} = C^\infty$ , or  $\mathcal{X} = C^\alpha$  (real-analytic maps), depending on the setup; to avoid confusion with the probabilistic use of  $\omega$  as an element of a sample space  $\Omega$  we use the notation  $\alpha$  as a superscript for real-analytic maps. In the notation we omit domain and range for function spaces, e.g.,  $f, g \in C^k$  is interpreted as  $f \in C^k(\mathcal{K}_x \times \mathcal{K}_y \times \mathcal{I}, \mathbb{R}^2)$  and in addition  $g \in C^k(\mathcal{K}_x \times \mathcal{K}_y \times \mathcal{I}, \mathbb{R}^1)$ .

The system (1) is written on the *slow time scale*  $\tau$  and can be re-written equivalently on the *fast time scale*  $t := \tau/\varepsilon$ . The *critical manifold* of (1) is

$$\mathcal{C}_0 := \{(x, y) \in \mathcal{K}_x \times \mathcal{K}_y \subset \mathbb{R}^3 : f(x, y, 0) = 0\}. \tag{2}$$

$\mathcal{C}_0$  can also be viewed as the algebraic constraint of the differential-algebraic *slow subsystem* obtained from (1) by taking the singular limit  $\varepsilon \rightarrow 0$  which yields

$$\begin{aligned} 0 &= f(x_1, x_2, y, 0), \\ \dot{y} &= g(x_1, x_2, y, 0). \end{aligned} \tag{3}$$

Alternatively, one may view  $\mathcal{C}_0$  as equilibrium points of the *fast subsystem*

$$\begin{aligned} \frac{dx}{dt} &= x' = f(x_1, x_2, y, 0), \\ \frac{dy}{dt} &= y' = 0. \end{aligned} \quad (4)$$

obtained as a singular limit from (1) on the time scale  $t$ . For a more detailed introduction to multiple time scale dynamics see [16]. The main assumptions for a generic *delayed* (or *dynamic*) *Hopf bifurcation* to occur in (1) are:

- (A1)  $\mathcal{C}_0$  is a real-analytic one-dimensional curve and we assume wlog that  $\mathcal{C}_0 = \{(x, y) \in \mathcal{K}_x \times \mathcal{K}_y : x_1 = 0, x_2 = 0\}$ ;
- (A2)  $\mathcal{C}_0$  is *normally hyperbolic* except at a single point, which we take without loss of generality to be the origin  $0 := (0, 0, 0)^\top \in \mathbb{R}^3$ ; more precisely, the matrix  $[D_x f](p, 0) =: J(p) \in \mathbb{R}^{2 \times 2}$  has eigenvalues  $\lambda_1(p) = a_1(p) - ib_1(p)$ ,  $\lambda_2(p) = a_2(p) + ib_2(p)$  such that  $a_{1,2}(p) \neq 0$  for every  $p \neq 0$ ,  $p \in \mathcal{C}_0$ ;
- (A3) at  $p = 0$  the fast subsystem has a Hopf bifurcation, i.e.,  $\lambda_{1,2}(0)$  is a complex conjugate pair of eigenvalues with nonzero imaginary part and we assume wlog that  $b_1(0) > 0$  and  $\text{sign}(a_{1,2}(x_1, x_2, y)) = \text{sign}(y)$ ;
- (A4) the fast subsystem Hopf bifurcation at  $p = 0$  is generic, i.e.,  $\frac{da_{1,2}}{dy}(0) \neq 0$  and the first Lyapunov coefficient is nonzero;
- (A5)  $g(0, 0, y, 0) \geq g_0 > 0$  for all  $y \in \mathcal{K}_y$  and some constant  $g_0 > 0$ .

By (A1), we may write the slow subsystem (3) as

$$\dot{y} = g(0, 0, y, 0). \quad (5)$$

Denote the solution of (5) with initial condition  $y(\tau_0) =: y_0$  by  $\xi(\tau)$ . The assumption (A5) guarantees that a trajectory crosses from the negative  $y$ -axis to the positive  $y$ -axis. Up to a time translation, we may always assume for each individual slow subsystem trajectory that  $\xi(0) = 0$ .

Trajectories  $\gamma_\varepsilon$  of the *full fast-slow system* (1) with initial conditions  $y_0 < 0$ ,  $y_0 = \mathcal{O}(1)$  as  $\varepsilon \rightarrow 0$ , starting sufficiently close to the  $y$ -axis first get attracted towards

$$\mathcal{C}_0^a := \{(x, y) \in \mathcal{K}_x \times \mathcal{K}_y : y < 0\} \cap \mathcal{C}_0.$$

Indeed,  $\mathcal{C}_0^a$  is normally hyperbolic *attracting* since by (A3) we have negative real parts of the eigenvalues for the linearization, so Fenichel's Theorem [9, 14] guarantees that there exists a *slow manifold*  $\mathcal{C}_\varepsilon^a$  which is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{C}_0^a$  and the flow on  $\mathcal{C}_\varepsilon^a$  converges to the flow on  $\mathcal{C}_0^a$  as  $\varepsilon \rightarrow 0$ ; see also Figure 2. By (A5), the slow dynamics on  $\mathcal{C}_\varepsilon^a$  guarantees that  $\gamma_\varepsilon$  approaches a neighbourhood of the origin. The behavior of  $\gamma_\varepsilon$  once it passes the Hopf bifurcation point and is near the *repelling* part

$$\mathcal{C}_0^r := \{(x, y) \in \mathcal{K}_x \times \mathcal{K}_y : y > 0\} \cap \mathcal{C}_0$$

of the critical manifold is characterized by the following classical result:

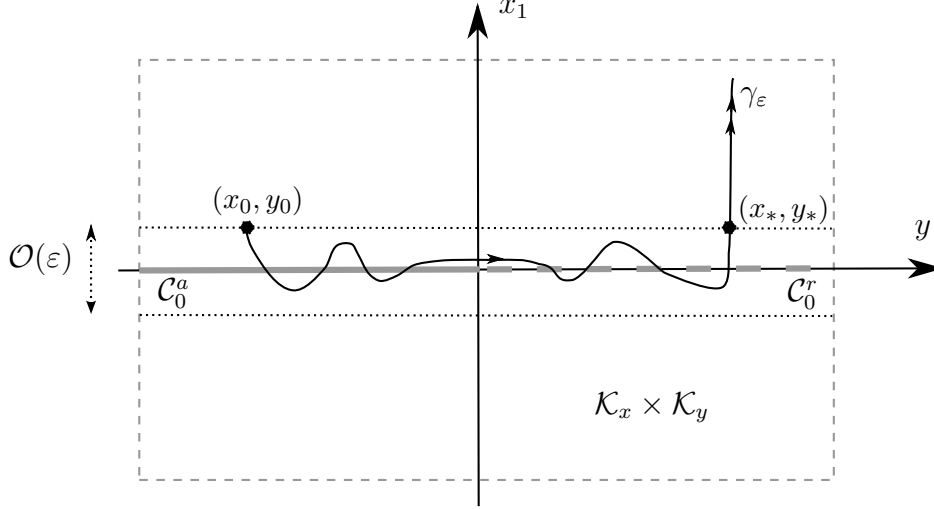


Figure 2: Sketch of situation for a delayed Hopf bifurcation. Projection onto  $(y, x_1)$ -coordinates. The domain  $\mathcal{K}_x \times \mathcal{K}_y$  is indicated as a dashed rectangle. A trajectory  $\gamma_\varepsilon$  is shown (black curve) getting first attracted to and then repelled from  $\mathcal{C}_0$ . The initial condition  $(x_0, y_0) = (x(\tau_0), y(\tau_0))$  is chosen so  $\tau_0$  is an asymptotic moment of falling while  $(x_*, y_*) = (x(\tau_*), y(\tau_*))$  marks the trajectory location for the asymptotic moment of jumping  $\tau_*$ , where  $\tau_*$  is also the delay time.

**Theorem 2.1** ([20, 21]). *Suppose (A1)-(A5) hold. Fix an initial time  $\tau_0 < 0$ . Assume  $(x_1(\tau_0), x_2(\tau_0), y(\tau_0))$  is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{C}_0^a$  with associated trajectory  $\gamma_\varepsilon(\tau)$ . Then, there exists  $\varepsilon_* > 0$  such that for all  $\varepsilon \in (0, \varepsilon_*]$ ,  $\gamma_\varepsilon$  is in an  $\mathcal{O}(\varepsilon)$ -neighborhood of  $\mathcal{C}_0^r$  for a delay time beyond the bifurcation point at  $y = 0$ .*

(D1) *Suppose  $f, g \in C^\alpha$  with complex analytic continuations in the  $(x, y)$ -variables in a neighborhood of the origin remaining smooth with respect to  $\varepsilon$ . Then  $\gamma_\varepsilon$  has a delay time  $\tau_* > 0$  where  $\tau_* = \mathcal{O}(1)$  as  $\varepsilon \rightarrow 0$ .*

(D2) *Suppose  $f, g \in C^\infty$ . Then the generic delay time is  $\sqrt{M(\varepsilon)\varepsilon|\ln \varepsilon|}$  where  $M(\varepsilon) \rightarrow +\infty$  monotonically as  $\varepsilon \rightarrow 0$ .*

(D3) *Suppose  $f, g \in C^l$  for  $l < \infty$ . Then the generic delay time is of the order  $\mathcal{O}(\varepsilon|\ln \varepsilon|)$  as  $\varepsilon \rightarrow 0$ .*

The important distinction between cases (D1) and (D2)-(D3) is that a long delay is observed in the analytic case and a short delay in the remaining cases. The genericity requirement in cases (D2)-(D3) is necessary to guarantee that  $\{y = 0\}$  is no longer an invariant manifold for the full system when  $\varepsilon > 0$  and we shall make this assumption from now on:

(A6)  $\mathcal{C}_0$  is not an invariant manifold for  $\varepsilon \in (0, \varepsilon_*]$ .

Furthermore, we are going to assume that  $\mathcal{O}(\cdot)$  is with respect to  $\varepsilon$  and omit  $\varepsilon \rightarrow 0$  from now on.

For case (D1), calculating  $\tau_*$  splits into two further cases.  $\tau_*$  is called an *asymptotic moment of jumping* of  $\gamma_\varepsilon$  if in an  $\mathcal{O}(\varepsilon|\ln \varepsilon|)$ -neighborhood of  $\tau$ , there is an interval  $[\tau_a, \tau_b]$  such that  $\gamma_\varepsilon(\tau_a)$  is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{C}_0$  and  $\gamma(\tau_b)$  is  $\mathcal{O}(1)$  separated from  $\mathcal{C}_0$ .  $\tau_*$  is called an *asymptotic moment of falling* if it is an asymptotic moment of jumping upon reversing time. Define the *complex phase*

$$\Psi(\tau) := \int_0^\tau \lambda_1(\xi(s)) \, ds.$$

Following [20, 21], we define the *entry/exit-map*  $\Pi : (-\infty, 0] \rightarrow [0, +\infty)$  by the requirement

$$\operatorname{Re}[\Psi(\tau)] = \operatorname{Re}[\Psi(\Pi(\tau))]. \quad (6)$$

Extending the domain of  $\tau$  from  $\mathbb{R}$  to  $\mathbb{C}$ , the pairs  $(\tau, \Pi(\tau))$  can be connected by arcs

$$\mathcal{L}_k = \{\tau \in \mathbb{C} \mid \operatorname{Re}[\Psi(\tau)] = k\} \subset \mathbb{C}.$$

which are level sets of  $\operatorname{Re}[\Psi(\tau)]$  for a given real number  $k \in \mathbb{R}$ .  $\operatorname{Re}[\Psi(\tau)]$  is sometimes called the *relief function*. Near  $\tau = 0$  the following conditions hold as consequences of (A1)-(A5):

(B1)  $\xi$  is analytic and  $f_1, f_2, g$  are analytic at points of the slow flow solution;

(B2)  $\lambda_{1,2}(\xi(\tau)) \neq 0$  and  $\lambda_1(\xi(\tau)) \neq \lambda_2(\xi(\tau))$ ;

(B3) no tangent to the curves  $\mathcal{L}_k$  is vertical.

(B1)-(B3) can fail far away from the Hopf bifurcation of the fast subsystem. Let  $\tau^-$  and  $\tau^+$  be the lower and upper bounds of endpoints of arcs  $\mathcal{L}_k$  for which (B1)-(B3) hold. Let  $\Gamma$  be the arc starting at  $\tau^-$  and ending at  $\tau^+$  on which  $\operatorname{Re}(\Psi(\tau))$  is constant. Denote the domain in the complex  $z$ -plane bounded by  $\Gamma$  and its conjugate arc  $\bar{\Gamma}$  by  $\mathcal{G}$ .

**Theorem 2.2** ([20, 21]). *Suppose  $\tau_0 \in (\tau^-, \tau^+)$  is an asymptotic moment of falling. Then  $\Pi(\tau_0)$  is an asymptotic moment of jumping, and on the interval*

$$(\tau_0 + K\varepsilon|\ln \varepsilon|, \Pi(\tau_0) - K\varepsilon|\ln \varepsilon|), \quad \text{for some fixed constant } K > 0$$

*the solution is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{C}_0$ . If  $\tau_0 < \tau^-$  then the solution generically remains  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{C}_0$  until  $\tau < \tau^+ - \delta(\varepsilon)$  where  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .*

The time  $\tau^+$  is called the *buffer time* and  $\xi(\tau^+)$  is the *buffer point*. Theorem 2.2 states that there are two cases: either the integrated linearized variational contraction and expansion balance to determine the asymptotic moment of jumping, or all trajectories leave near the buffer point.

## 3 Random Delayed Hopf Bifurcation

### 3.1 Basic Setup

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Consider the random (2,1)-fast-slow system

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y, \varepsilon), \\ \dot{y} &= g(x, y, \varepsilon), \end{aligned} \quad (7)$$

with initial condition  $(x(\tau_0), y(\tau_0)) = (x_0(\omega), y_0(\omega))$ . We are going to use as a solution concept for (7) *sample function solutions* [26]. Suppose (7) has a delayed Hopf bifurcation satisfying assumptions (A1)-(A6) for every  $\omega \in \Omega$ . Next, one may divide the vector field by  $g$  due to the assumption (A5) and re-scale time to obtain

$$\begin{aligned}\varepsilon \dot{x} &= \tilde{f}(x, y, \varepsilon), \\ \dot{y} &= 1,\end{aligned}\tag{8}$$

where we are going to drop the tilde from  $f$  in this section and focus on studying the system (8) satisfying (A1)-(A6) with initial condition  $(x(\tau_0), y(\tau_0)) = (x_0(\omega), y_0(\omega))$ . This makes the slow subsystem particularly simple so

$$\dot{y} = 1, \quad \Rightarrow \quad y(\tau) = (\tau - \tau_0) + y_0(\omega).\tag{9}$$

We make the standard assumption that for each individual trajectory of the slow subsystem we require  $\xi(0) = 0$ , which implies that  $\tau_0 = \tau(\omega)$  becomes a random variable with the same distribution as  $y_0(\omega)$ . Of course, we can calculate from the distribution of the asymptotic moment of jumping  $\tau_*(\omega) = \Pi(\tau(\omega))$  the distribution of  $y_*(\omega) := y(\tau_*(\omega))$  just using (9).

As a first step, we are only interested in the dynamic bifurcation effect in the slow coordinate  $y$  and not in the precise location of the (oscillatory) fast variables  $x$ . So we take  $(x(0), y(0)) = (x_0, y_0(\omega))$  with

$$\mathbb{P}(y \leq y_0 \leq y + dy) = \mu_0(y_0), \quad \text{supp}(\mu_0) \subset (-\infty, -\kappa_\mu],\tag{10}$$

where  $\mu_0$  is a probability measure and  $\kappa_\mu > 0$  is some given sufficiently small constant as we are not interested in initial conditions that do not undergo at least a certain delay. In particular, we can always make  $\varepsilon$  sufficiently small to ensure that  $x_0$  is not only deterministic but we also have  $x(t) = \mathcal{O}(\varepsilon)$  after a short transient time  $t$  since  $\mathcal{C}_0^a$  is globally attracting for each fixed  $y < 0$ . Therefore, we shall just assume  $x_0 = \mathcal{O}(\varepsilon)$  and  $(x_0, y_0(\omega)) \notin \mathcal{C}_\varepsilon^a$  directly; see Figure 2. If  $\mu_0$  admits a probability density, then we denote the density by  $p_0$ . The probability measure associated to  $\tau_*$  will be denoted by  $\mu_*$  and if it has a density we call it  $p_*$ .

### 3.2 Orthogonal Transformation of Uncertainty

The first situation we are going to analyze is what could be called simple passage (or orthogonal transformation) of uncertainty, i.e., of the probability density of initial conditions under delayed bifurcation by reflection and/or translation.

**Theorem 3.1.** *Suppose  $\mu_0$  has compact support. Then there exists  $f \in C^\alpha$  such that (A1)-(A6) hold and  $\mu_*(y) = \mu_0(-y)$ .*

*Proof.* Since  $\mu_0$  has compact support it follows that we can restrict to  $\mathcal{K}_y$  compact. It suffices to find  $f$  such that: (A1)-(A6) hold, the buffer time  $\tau^+$  can be made large enough to move any buffer points outside  $\mathcal{K}_x \times \mathcal{K}_y$ , and  $\tau_* = -\tau_0$ . Consider

$$f(x_1, x_2, y, \varepsilon) = \begin{pmatrix} cyx_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + cyx_2 - x_1(x_1^2 + x_2^2) \end{pmatrix} + \mathcal{O}(\varepsilon)\tag{11}$$

for some constant  $c > 0$  to be chosen below, and select analytic higher-order  $\mathcal{O}(\varepsilon)$ -terms such that  $\mathcal{C}_0 \neq \mathcal{C}_\varepsilon$  which yields (A6). Since  $f$  is just to leading-order the normal form of a generic Hopf bifurcation with parameter  $cy$ , it easily follows that (A1)-(A4) are satisfied and (A5) trivially holds for  $\dot{y} = 1$ . Next, we want to analyze the upper bounds on the buffer time imposed by the conditions (B1)-(B3). One calculates that  $\lambda_{1,2}(s) = cs \mp i$  so

$$\Psi(\tau) = \int_0^\tau cs - i \, ds = \frac{c}{2}\tau^2 - i\tau.$$

Hence, letting  $\tau = u + iv$  one gets

$$\operatorname{Re}(\Psi(\tau)) = \frac{c}{2}(u^2 - v^2) + v := U(u, v).$$

We start with the upper bound imposed by (B3). Vertical tangents to the level sets  $\mathcal{L}_k = \{U(u, v) = k\}$  appear if  $\frac{\partial U}{\partial v} = -cv + 1 = 0$ , i.e., for  $v = 1/c$ . Level curves connecting from the vertical tangency point  $(u_1, 1/c)$  to a point  $(u_2, 0)$  have to satisfy

$$\frac{c}{2}(u_1^2 - c^{-2}) + \frac{1}{c} = k, \quad \text{and} \quad \frac{c}{2}u_2^2 = k.$$

The level curves  $\mathcal{L}_k$  delimiting  $\mathcal{G}$  in the upper half-plane are given by  $k = 1/(2c)$  so  $u_2 = \pm 1/c$ . This implies that (B3) yields an upper bound on the buffer time given by  $1/c$ . Regarding (B2), it is easy to see that  $\lambda_1 \neq \lambda_2$  and  $\lambda_1(s) = 0$  if and only if  $s = i/c$  or  $v = 1/c$ . Therefore, (B2) leads to the same upper bound on the buffer time as the condition (B3). Lastly, (B1) does not yield any upper bound on  $\tau^+$ . Therefore, we find  $\tau^+ = 1/c$  and we can move any buffer point outside of a given compact region  $\mathcal{K}_x \times \mathcal{K}_y$  upon decreasing  $c$ . Lastly, we have to check that  $\tau_* = -\tau_0$ . Again one may just calculate that

$$\operatorname{Re}(\Psi(\tau_0)) = \operatorname{Re}(\Psi(\tau_*)) \quad \Leftrightarrow \quad \frac{c}{2}\tau_0^2 = \frac{c}{2}\tau_*^2$$

since the start and end times must be real-valued. Upon using that  $\tau_* > \tau_0$  and  $c > 0$  we may conclude that  $\tau_* = -\tau_0$ .  $\square$

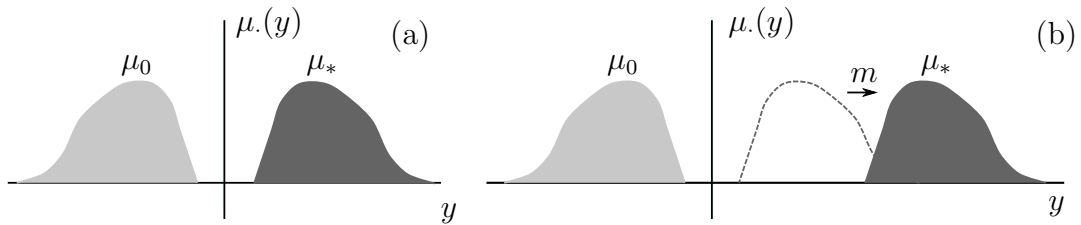


Figure 3: Sketch of the situation for orthogonal transformations. (a) Illustration of Theorem 3.1 where  $\mu_*(y) = \mu_0(-y)$  is obtained via reflection. The initial density is shown in light grey and the transported one in dark grey. (b) The situation involving reflection (dashed density) and a nontrivial shift by  $m \neq 0$  is not possible for analytic vector fields according to Theorem 3.2.

The proof of Theorem 3.1 carries over verbatim if higher-order nonlinear perturbations are added to the Hopf normal form (3.1); see Figure 3(a) for an illustration. However, not all natural transformations preserving the shape of the  $\mu_0$  are allowed as the next result, quite surprisingly, shows.



**Theorem 3.2.** Fix  $m \in \mathbb{R} - \{0\}$ , suppose  $\mu_0$  has compact support  $\text{supp}(\mu_0) = \mathcal{I}_\mu$  containing at least one accumulation point, and  $\text{supp}(\mu_0(\cdot + m)) \subset (-\infty, 0)$ . Then there does not exist  $f \in C^\alpha$  such that (A1)-(A6), (B1)-(B3) hold and

$$\mu_*(y) = \mu_0(-y + m).$$

*Proof.* We argue by contradiction and suppose that an analytic vector field  $f$  exists satisfying (A1)-(A6), (B1)-(B3) such that  $\mu_*(y) = \mu_0(-y + m)$  for some positive  $m > 0$ ; note that this situation corresponds to reflecting and shifting the initial condition  $y$ -distribution. Using (B1)-(B3) and Theorem 2.2 for the case of no buffer points it follows that

$$\text{Re}(\Psi(-\tau_0 + m)) - \text{Re}(\Psi(\tau_0)) = 0$$

for all  $\tau_0 \in \mathcal{I}_\mu$ . As above let  $\lambda_1(s)$  denote the eigenvalue in the definition of  $\Psi$ . Since (B2) is always assumed to hold independent of the point  $p$ , it follows that the discriminant of the Jacobian  $J(p)$  does not vanish. The discriminant must be negative to get complex conjugate eigenvalues. Therefore,  $\lambda_1(s) = a_1(s) - ib_1(s)$  where  $a_1(s)$  is real-analytic as a function of  $s \in \mathbb{R}$  and

$$\text{Re}(\Psi(-\tau_0 + m)) - \text{Re}(\Psi(\tau_0)) = \int_{\tau_0}^{-\tau_0 + m} a_1(s) \, ds =: A_1(\tau_0).$$

Since  $a_1$  is real-analytic it follows that  $A_1(\tau_0)$  is real-analytic.  $A_1(\tau_0)$  vanishes on  $\mathcal{I}_\mu$  which contains an accumulation point. Extending  $A_1(\tau_0)$  to a sufficiently small neighbourhood of  $\mathcal{I}_\mu$  into the complex plane we may apply the principle of permanence and conclude that  $A_1(\tau_0)$  vanishes also at  $\tau_0 = 0$ . This implies

$$\int_0^m a_1(s) \, ds = 0$$

and so, since  $m > 0$  and  $a_1(s) > 0$  for  $s > 0$  by (B2), we obtain a contradiction.  $\square$

The last result shows that there is some rigidity in the way uncertainty can be transported for analytic vector fields without buffer points and suitable uniform eigenvalue configurations; see also Figure 3(b) and Section 3.5.

### 3.3 Random-to-Deterministic Mapping

In Section 3.2 we have considered the case when uncertainty gets just mapped via orthogonal transformations (translation, reflection). In this section, we address the other extreme case and study when we obtain a deterministic, or at least almost deterministic, output after passing a delayed bifurcation.

**Proposition 3.3.** Given  $\mu_0$  satisfying (10), there exists  $f \in C^\alpha$  such that (A1)-(A6) hold and  $\mu_*(y) = \delta_{y^+}(y)$ , i.e., the output is a delta-distribution located at some  $y^+ > 0$ .

*Proof.* As in the proof of Theorem 3.1 we select (11). By (10) the support of  $\mu_0$  is contained in  $(-\infty, -\kappa_\mu]$  for some fixed positive  $\kappa_\mu > 0$ . Recall from the proof of Theorem 3.1 that the buffer point is given by  $\tau^+ = 1/c$ , where  $c > 0$  is the parameter in the vector field (11). Making  $c$  sufficiently small we can guarantee that  $\tau_* > 1/c$  so all trajectories with initial conditions sampled from  $\mu_0$  jump at the buffer point.  $\square$

Proposition 3.3 shows that in the singular limit  $\varepsilon = 0$  any uncertainty in the initial condition disappears. In fact, one may even do slightly better for initial distributions with compact support and exhibit examples for any target  $\delta$ -distribution.

**Theorem 3.4.** *Let  $\mu_0$  have compact support and satisfy (10). Furthermore, fix any  $y^+ > 0$ . Then there exists an analytic vector field  $f$  satisfying (A1)-(A6) such that*

$$\mu_*(y) = \delta_{y^+}(y).$$

*Proof.* Consider the modified Hopf normal form

$$f(x_1, x_2, y, \varepsilon) = \begin{pmatrix} e^{-ay}yx_1 - bx_2 - x_1(x_1^2 + x_2^2) \\ bx_1 + e^{-ay}yx_2 - x_1(x_1^2 + x_2^2) \end{pmatrix} + \mathcal{O}(\varepsilon) \quad (12)$$

with parameters  $a, b > 0$  and suitable higher-order analytic  $\mathcal{O}(\varepsilon)$ -terms such that  $\mathcal{C}_0 \neq \mathcal{C}_\varepsilon$ . A direct calculation shows that  $\lambda_{1,2}(s) = e^{-as}s \mp bi$ . First, we are going to investigate the locations of buffer points. One has for  $\tau = u + iv$  with  $u, v \in \mathbb{R}$  that

$$\operatorname{Re}(\Psi(\tau)) = bv - \frac{e^{-au}((au + 1)\cos(av) + av\sin(av))}{a^2}.$$

Upon increasing  $a > 0$  we can ensure that (B3) is satisfied in a region  $\mathcal{G}$  delimited by two arcs  $\Gamma$  and  $\bar{\Gamma}$  such that  $\mathcal{G}$  is inside a region

$$\{u + iv \in \mathbb{C} : u \in [-u^-, u^+], [-v^-, v^+]\}$$

where  $u^-, v^-, v^+ > 0$  are fixed. Since the level curves of  $\operatorname{Re}(\Psi(\tau))$  become almost horizontal in the positive half-plane in the limit  $a \rightarrow +\infty$ , it follows that  $u^+ \rightarrow +\infty$  as  $a \rightarrow +\infty$ . Therefore, (B3) gives no upper bound for the buffer time  $\tau^+$  if  $a$  is sufficiently large. Clearly, (B1) always holds. Regarding (B2), observe that  $\lambda_1(s^+) = 0$  if and only if

$$ib = e^{-as^+}s^+. \quad (13)$$

The solutions  $s^+ = s^+(b)$  of the transcendental equation (13) for  $s^+ \in \mathbb{C}$  satisfy

$$\lim_{b \rightarrow 0} |s^+(b)| = 0 \quad \text{and} \quad \lim_{b \rightarrow +\infty} |s^+(b)| = +\infty$$

for fixed  $a > 0$ . Hence, once  $a > 0$  has been fixed we can use  $b > 0$  to get any buffer time  $\tau^+$ . Next, we show that  $a > 0$  can indeed be chosen and fixed so that any trajectory with initial condition sampled from  $\mu_0$  does reach the buffer time before escaping. One has for real values of  $\tau$  and  $\tau_0$  that

$$\operatorname{Re}(\Psi(\tau_*)) - \operatorname{Re}(\Psi(\tau_0)) = 0 \quad \Leftrightarrow \quad (a\tau_* + 1)e^{a\tau_0} = e^{a\tau_*}(a\tau_0 + 1)$$

Since  $\mu_0$  has compact support contained in  $(-\infty, -\kappa_\mu]$  for some  $\kappa_\mu > 0$  there exists a large  $a > 0$  such that  $\tau_* = +\infty$  for all  $\tau_0 \leq -\kappa_\mu$ . Therefore, if we select  $a > 0$  sufficiently large and then select  $b > 0$  we can achieve any finite buffer time. All trajectories sampled from  $\mu_0$  jump at this prescribed buffer time, respectively the prescribed buffer point.  $\square$

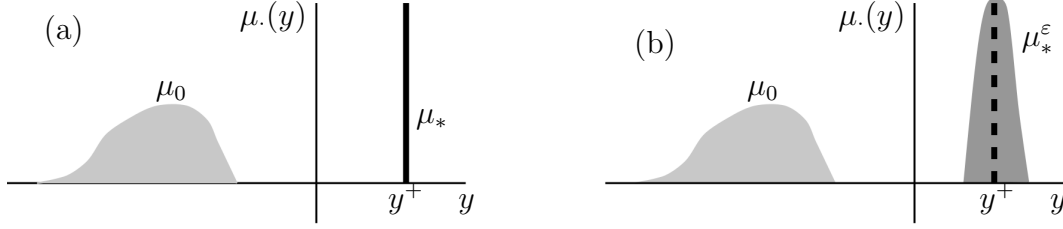


Figure 4: Sketch of the transformation to an almost deterministic output. (a) Illustration of Theorem 3.4 in the singular limit  $\varepsilon = 0$ . The initial density is shown in light grey and the transported delta mass as a solid black bar. (b) The same situation as in (a) just for  $0 < \varepsilon \ll 1$  as stated in Corollary 3.6; the dashed line indicates the singular limit distribution and the dark grey is the transformed density.

The main insight in the last proof is that an asymmetric strength of the attracting and repelling eigenvalues can be used to make the final jump time  $\tau_*$  calculated from entry-exit map large using one parameter. The second parameter is then used to create and move a buffer point  $\tau^+$  which leads to an escape; see also Figure 4(a) for an illustration of Theorem 3.4.

It is clear that the results in Proposition 3.3 and Theorem 3.4 are not quite what would be observed in practice in a numerical or experimental setup as one has to consider  $0 < \varepsilon \ll 1$  instead of  $\varepsilon = 0$  as shown in Figure 4(b). To analyze this case we need some preliminary considerations. Define the set

$$\mathcal{T}(h) := \{(x, y) \in \mathbb{R}^3 : \|x\| \leq h\varepsilon\}$$

and fix  $h > 0$  always so that

$$\mathcal{C}_\varepsilon^a, \mathcal{C}_\varepsilon^r \subset \mathcal{T}(h) \quad \text{and} \quad \partial\mathcal{T}(h) \cap (\mathcal{C}_\varepsilon^a \cup \mathcal{C}_\varepsilon^r) = \emptyset,$$

where  $\partial\mathcal{T}(h)$  denotes the boundary of  $\mathcal{T}(h)$ . Let  $p_0^\varepsilon(y)$  be a probability density of initial  $y$ -coordinate conditions with the same support condition as in (10) and fix some  $x_0$  such that  $(x_0, y_0(\omega)) \in \partial\mathcal{T}(h)$ , where  $y_0(\omega)$  is sampled from  $p_0^\varepsilon(y)$ . Let  $y_*(\omega)$  denote the  $y$ -coordinate of the point in  $\partial\mathcal{T}(h)$ , where a trajectory of (8) starting at  $y_0(\omega)$  first leaves  $\mathcal{T}(h)$ . Denote the associated probability density of  $y_*(\omega)$  by  $p_*^\varepsilon(y)$ .

**Lemma 3.5.** *Suppose  $f, p_0^\varepsilon \in C^r$  for some  $r \in \mathbb{N}_0$ ,  $r = \infty$ , or  $r = \alpha$ , then  $p_*^\varepsilon \in C^r$ .*

*Proof.* The result for  $r \in \mathbb{N}_0$  and  $r = \infty$  follows from the classical theory of continuous/differentiable and smooth dependence of solutions of ODEs on initial conditions [6]. In fact, one may also prove that for analytic vector fields solutions depend analytically on initial data [24, Sec.C.3].  $\square$

**Corollary 3.6.** *Suppose  $f \in C^\alpha$ ,  $p_0^\varepsilon \in C^\infty$  for all  $\varepsilon \in [0, \varepsilon_*]$  for some sufficiently small  $\varepsilon_* > 0$ , and  $p_0^\varepsilon$  has compact support satisfying (10). Furthermore, fix any  $y^+ > 0$ . Then there exists an analytic vector field  $f$  satisfying (A1)-(A6) such that*

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} p_0^\varepsilon(y) w(y) dy = w(y^+) \quad (14)$$

for all  $w \in C_c^\infty$  (smooth functions with compact support).

*Proof.* This is just a combination of Theorem 2.2, Theorem 3.4, and Lemma 3.5.  $\square$

Therefore, we observe in practice the convergence to a  $\delta$ -distribution via an approximation to the identity. It is clear that also a modification of Corollary 3.6 holds which uses the assumptions of Proposition 3.3 instead the ones from Theorem 3.4.

### 3.4 Mixtures

The case of purely deterministic output distribution  $\mu_* = \delta_{\tau_+}$  is already very interesting for applications. In this section, we study the case when  $\mu_0$  is neither symmetrically reflected nor mapped to a completely deterministic distribution  $\mu_*$  as shown in Figure 5.

**Theorem 3.7.** *Suppose  $f \in C^\alpha$  and assumptions (A1)-(A6), as well as (B1)-(B3) hold up to a given buffer time  $\tau_+ \in (0, \infty)$ . Then  $\mu_*$  is a mixture measure*

$$\mu_* = \rho_{*,1} \delta_{\tau_+} + \rho_{*,2} \mu_{*,2}, \quad (15)$$

for some  $\rho_{*,1} + \rho_{*,2} = 1$  and a probability measure  $\mu_{*,2}$ . If  $\tau_* = -\tau_0$  for  $\tau_* < \tau_+$  holds and  $\mu_0$  satisfying (10) has density  $p_0$  then

$$\rho_{*,1} = \int_{-\infty}^{\tau_-} d\mu_0(s), \quad \rho_{*,2} = \int_{\tau_-}^0 d\mu_0(s), \quad d\mu_{*,2}(s) = \mathbf{1}_{\{0 \leq s < \tau_+\}} d\mu_0(-s). \quad (16)$$

*Proof.* The result (15) follows from the existence of a buffer time  $\tau_+$  since all times  $\tau < \tau_-$  satisfy  $\Pi(\tau) = \tau_+$  yielding a delta-distribution at  $\tau_+$ . The special case (16) holds since  $-\tau_0 = \tau_*$  for times not reaching the buffer time yields  $\mu_*(s) = \mu_0(-s)$ , and the weights  $\rho_{*,1}$  and  $\rho_{*,2}$  are just computed from the probability which points reach the buffer time and which ones do not.  $\square$

Although the assumption  $-\tau_0 = \tau_*$  seems quite special, it should be noted that this is precisely the situation which happens for the standard Hopf bifurcation normal form; cf. Theorem 3.1 and the vector field (11).

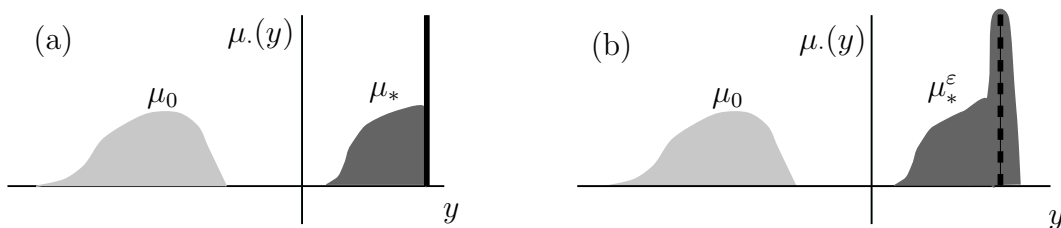


Figure 5: Sketch of the mixture situation from Theorem 3.7. The notational and grey-shading conventions are as in Figures 3-4. (a) Singular limit  $\varepsilon = 0$  leading to the mixture of a delta-mass at the buffer point and a remaining part calculated via the entry/exit-map. (b) The same situation as in (a) just for  $0 < \varepsilon \ll 1$ .

It is interesting to compute with a few classical initial time distributions for the Hopf normal form case. To simplify the notation we are going to define

$$m_{0,q} := \int_{-\infty}^{\infty} s^q d\mu_0(s), \quad m_{*,q} := \int_{-\infty}^{\infty} s^q d\mu_*(s), \quad (17)$$

for  $q \in \mathbb{N}$  as the  $q$ -th moments of  $\mu_0$  and  $\mu_*$ . We are going to assume that the moments do exist.

**Corollary 3.8.** *Suppose  $f \in C^\alpha$  and satisfies assumptions (A1)-(A6), as well as (B1)-(B3) up to a given buffer time  $\tau_+ \in (0, \infty)$ . Furthermore, assume  $\tau_* = -\tau_0$  for  $\tau_* < \tau_+$  holds and  $\mu_0$  satisfying (10) has density  $p_0$  then*

$$m_{*,q} = \left( \int_{\tau_-}^0 p_0(s) ds \right) \left( \int_0^{\tau_+} s^q p_0(-s) ds \right) + \tau_+^q \int_{-\infty}^{\tau_-} p_0(s) ds. \quad (18)$$

**Proposition 3.9.** *Suppose  $p_0$  is a uniform density with support in  $[-b, -a]$  for  $b > a > 0$  and the assumptions of Corollary 3.8 hold. Then three cases occur*

$$(U1) \ a \geq \tau_+ : m_{*,q} = (\tau_+)^q,$$

$$(U2) \ a < \tau_+ \leq b : m_{*,q} = \frac{(b-\tau_+)(\tau_+)^q}{b-a} + \frac{(\tau_+-a)((\tau_+)^{q+1}-a^{q+1})}{(b-a)(q+1)},$$

$$(U3) \ b \leq \tau_+ : m_{*,q} = \frac{b^{q+1}-a^{q+1}}{(b-a)(q+1)}.$$

*Proof.* Only the case (U2) is interesting, the other two cases are trivial. Since  $\tau_* = -\tau_0$  for  $\tau_0 > \tau_-$  we also must have  $-\tau_- = \tau_+$ . Therefore, we find

$$\rho_{*,1} = \int_{-b}^{-\tau_+} \frac{1}{b-a} ds = \frac{b-\tau_+}{b-a}, \quad \rho_{*,2} = \int_{-\tau_+}^{-a} \frac{1}{b-a} ds = \frac{\tau_+-a}{b-a},$$

and so using Theorem 3.7 and calculating

$$\int_0^{\tau_+} s^q p_0(-s) ds = \frac{b^{q+1}-a^{q+1}}{(b-a)(q+1)}$$

yields the result. □

We also refer to cases, such as (U2), to the *full mixture* case, i.e., when  $\rho_{*,1} \neq 0$  and  $\rho_{*,2} \neq 0$ . If  $\rho_{*,1} = 0$  or  $\rho_{*,2} = 0$ , such as for (U1) or (U3), we refer to the situation as *singular mixture*.

**Proposition 3.10.** *Suppose  $p_0$  is an exponential density with support in  $(-\infty, -a]$  for  $a > 0$  with rate  $\beta^{-1}$ , i.e.,  $p_0(s) = \beta^{-1}e^{(s+1)/\beta}$  and the assumptions of Corollary 3.8 hold. Then two cases occur*

$$(E1) \ a \geq \tau_+ : m_{*,q} = (\tau_+)^q,$$

(E2)  $a < \tau_+$ : let  $\Gamma(z_1, z_2) = \int_{z_2}^{\infty} s^{z_1-1} e^{-s} ds$  be the incomplete gamma-function, then

$$m_{*,q} = e^{(a-\tau_+)/\beta} (\tau_+)^q + e^{1/\beta} \beta^q (1 - e^{(a-\tau_+)/\beta}) \left( \Gamma\left(q+1, \frac{a}{\beta}\right) - \Gamma\left(q+1, \frac{\tau_+}{\beta}\right) \right).$$

*Proof.* The only minor difference to the type of calculation in the proof of Proposition 3.9 is that the integral  $\int_0^{\tau_+} s^q p_0(-s) ds$  is slightly more complicated and can be easily re-written in terms of incomplete gamma-functions.  $\square$

In principle, one can now also do many other types of calculations for given classical probability measures  $\mu_0$ . Furthermore, similar smooth approximation results as Corollary 3.6 hold for mixture cases but we are not going to state them explicitly here.

### 3.5 Distribution Transformation

In addition to rigid transformations, random-to-deterministic mappings, and mixture distributions, one may also ask under which conditions we could obtain a particular target distribution. The question is, given two probability measures  $\mu_{\text{in}}$  and  $\mu_{\text{out}}$ , does there exist a vector field such that  $\mu_0 = \mu_{\text{in}}$  and  $\mu_{\text{out}} = \mu_*$ ? We have already seen in Theorem 3.2 that this question is nontrivial. We make the following assumptions for the problem setup:

- (M1) suppose  $\mu_0$  is given and has density  $p_0$ ,
- (M2) there exists a mapping  $\Pi$ , which is invertible and analytic on  $\text{supp}(p_0)$ , such that  $\Pi(\tau_0) = \tau_*$ .

Assumption (M1) on the existence of a density is made for notational convenience; it can be slightly relaxed to the accumulation point condition as in Theorem 3.2. Assumption (M2) implies  $p_*(r) = (p_0 \circ \Pi^{-1})(r) \frac{d\Pi^{-1}}{dr}(r)$ . Dropping analyticity in (M2) leads to a relatively trivial problem as Theorem 2.1 implies that the number of target measures  $\mu_*$  is extremely restricted in the case  $f \notin C^\alpha$ . We are going to use the notation

$$\Pi(s) = \sum_{k=0}^{\infty} \pi_k s^k \quad (19)$$

for the convergent power series of the map  $\Pi$ . We are going to show that for certain classes of given analytic maps  $\Pi$  a certain necessary condition for the existence of an analytic vector field  $f$ , satisfying (A1)-(A6) and (B1)-(B3), can be based upon on the classical theory of infinite matrices. For infinite-dimensional matrices with countable indices we use the notation  $M = (m_{ij})_{i,j \in \mathbb{N}}$  and denote by  $\text{Id}$  the matrix with entries given by the Kronecker delta  $\delta_{ij}$ .  $M$  is lower-diagonal if  $m_{ij} = 0$  for all  $j > i$ .

**Theorem 3.11.** *A necessary condition for the existence of  $f \in C^\alpha$  without buffer points such that (M1)-(M2), (A1)-(A6), and (B1)-(B3) hold is the existence of an infinite vector  $v := (v_1, v_2, \dots)^\top$  with  $v_k \in \mathbb{R}$  such that  $v_k \neq 0$  for some  $k \in \mathbb{N}$ , and  $v$  satisfies a linear system*

$$(M - \text{Id})v = 0 \quad (20)$$

for a matrix  $M$ , and  $M$  is computable recursively from  $\{\pi_k\}_{k \in \mathbb{N}_0}$  if  $\pi_0 \neq 0$ .

*Proof.* As in the proof of Theorem 3.2, we know that the discriminant of the Jacobian along the critical manifold is negative so that  $\lambda_1 = a_1 + ib_1$  where  $a_1 = a_1(s)$  is real-analytic. Furthermore, since there are no buffer points, we must have

$$0 = \operatorname{Re}(\Psi(\tau_*)) - \operatorname{Re}(\Psi(\tau_0)) = \int_{\tau_0}^{\tau_*} a_1(s) \, ds = A_1(\tau_*) - A_1(\tau_0) \quad (21)$$

where  $A_1$  is obtained via term-by-term integration of  $a_1$ . Clearly,  $A_1$  is real-analytic as well and we use the notation  $A_1(s) = \sum_{k=2}^{\infty} v_{k-1} s^k$ ; note the vanishing of the eigenvalue  $a_1(0) = 0$  in (A3) implies the particular form of the power series of  $A_1$ . Clearly, (21) is a necessary condition for the existence of  $f$ , which can be re-written as

$$\begin{aligned} A_1(\tau_*) - A_1(\tau_0) &= A_1(\Pi(\tau_0)) - A_1(\tau_0) \\ &= \sum_{k=2}^{\infty} v_{k-1} \left( \sum_{j=0}^{\infty} \pi_j s^j \right)^k - \sum_{k=2}^{\infty} v_{k-1} s^k = 0. \end{aligned} \quad (22)$$

Since  $\mu_0$  has a density  $p_0$ , it follows that the domain of  $\lambda_1$ , and hence the domain of  $a_1$  and  $A_1$ , has an accumulation point. Therefore, the equality (22) holds if and only if the coefficients of each power  $s^k$  vanish identically. We can re-write (22) as the solution of a linear system with an infinite matrix  $M$  so that  $(M - \operatorname{Id})v = 0$ . It is easy to see that  $M$  is computable recursively from  $\pi_j$ ; indeed, if we let

$$\Pi(s)^k = \left( \sum_{j=0}^{\infty} \pi_j s^j \right)^k =: \sum_{j=0}^{\infty} \pi_{j,k} s^j \quad (23)$$

then  $\pi_{0,k} = \pi_0^k$  and  $\pi_{j,k} = \frac{1}{j\pi_0} \sum_{l=1}^j [(k+1)l - j] \pi_l \pi_{j-l,k}$  which follows from differentiating (23) with respect to  $s$  and re-arranging terms.  $\square$

It may seem that computing  $v$  by solving (20) may also be sufficient since one can just use the coefficients  $v_j$  to get the required analytic eigenvalue function  $a_1(s)$  using the same trick as in the proof of Theorem 3.4 replacing  $e^{-ay}y$  in (12) by a more general analytic function of  $y$ . However, the problem is that  $\sum_{k=0}^{\infty} v_k s^k$  may not be a convergent power series on the required domain of definition.

At first, it may seem natural to adopt an operator-theoretic viewpoint for  $M$ . Let  $w : \mathbb{N} \rightarrow (0, +\infty)$  be a weight function and consider

$$l^p(\mathbb{N}, w) := \left\{ v = (v_k)_{k \in \mathbb{N}} : \|v\|_{p,w} := \left( \sum_{k=1}^{\infty} |v_k|^p w(k) \right)^{1/p} < \infty \right\} \quad (24)$$

for  $p \in [1, +\infty)$ . The next result shows why an operator-theoretic viewpoint leads to substantial complications for arbitrary analytic maps  $\Pi$ .

**Lemma 3.12.** *Given any weight function  $w$  such that  $(w(k))_{k \in \mathbb{N}} \in l^1(\mathbb{N}, 1)$  and fix any  $p \in [1, +\infty)$ , then there exists an analytic map  $\Pi$  such that  $M$  does not map  $l^p(\mathbb{N}, w)$  into itself.*

*Proof.* Observe that the first row of  $M$  is given by

$$M_{1,\cdot} = (\pi_2, \dots, (M_{1,k-1} + k - 1)\pi_0^{k-2}\pi_1^2 + k\pi_0^{k-1}\pi_2, (M_{1,k} + k)\pi_0^{k-1} + (k+1)\pi_0^k\pi_2, \dots).$$

Now select  $v = (1, 1, 1, \dots)$  which is clearly in  $l^p(\mathbb{N}, w)$  as  $(w(k))_{k \in \mathbb{N}} \in l^1(\mathbb{N}, 1)$ . Therefore, we have

$$(Mv)_1 = \sum_{k=1}^{\infty} M_{1,k}$$

which diverges if we select suitable a suitable map  $\Pi$ , say for example  $\pi_{0,1,2} = 1$ , and so  $(Mv) \notin l^p(\mathbb{N}, w)$ .  $\square$

Similar results hold for other function spaces, i.e.,  $M$  is not tractable using the classical theory of bounded operators if  $\Pi$  is arbitrary. Many different restrictions for  $\Pi$  are possible but a natural assumption is  $\pi_0 = 0$  as this corresponds to the condition  $\Pi(0) = 0$  which should be imposed in the limit  $\kappa_\mu \rightarrow 0$ , i.e., if the support of  $p_0$  limits onto  $s = 0$ , since any vector field maps the initial condition to itself if no time has elapsed.

**Lemma 3.13.** *If  $\pi_0 = 0$  then  $M$  is lower-diagonal with entries  $m_{ii} = \pi_1^{i+1}$ .*

*Proof.* Recall that  $M$  is constructed by collecting terms of different orders of  $s^l$  for  $l \in \{2, 3, 4, \dots\}$  from the expression

$$\sum_{k=2}^{\infty} v_{k-1} \left( \sum_{j=0}^{\infty} \pi_j s^j \right)^k = \sum_{k=1}^{\infty} v_k \left( \sum_{j=1}^{\infty} \pi_j s^j \right)^{k+1}$$

where we have used  $\pi_0 = 0$ . In particular,  $m_{ij}$  is can be nonzero if and only if  $v_j$  appears in the term collected for order  $l = i + 1$ . Therefore, fixing any  $i$  the largest index where a possible nonzero entry  $m_{ij}$  occurs is for  $j = i$ . In fact, it is also easy to see that the only term arising on the diagonal is  $\pi_1^{i+1}$ .  $\square$

Even though  $M$  is lower-diagonal, its entries still grow via certain multinomial coefficients. Therefore,  $M$  is not be a bounded operator for many  $\Pi$ . The following result even shows that it will be impossible to find an exact solution  $v$  in many cases without imposing additional conditions.

**Proposition 3.14.** *If  $\pi_0 = 0$  and  $\pi_1 \neq \pm 1$ , then  $(M - \text{Id})v = 0$  if and only if  $v = (0, 0, 0, \dots)^\top$ .*

*Proof.* From Lemma 3.13 we know that  $m_{11} = \pi_1^2$ . The first row of  $(M - \text{Id})v = 0$  yields

$$0 = v_1(m_{11} - 1) = v_1(\pi_1^2 - 1) \quad \Leftrightarrow \quad v_1 = 0$$

as  $\pi_1^2 \neq 1$ . A direct process by mathematical induction on the rows yields the result.  $\square$

In general, it is no problem to impose the condition  $\pi_1 = 1$  or  $\pi_1 = -1$  as we only want to match a pair of densities  $p_0, p_*$  via  $\Pi$  which have supports outside of the compact set  $[-\kappa_\mu, \Pi(-\kappa_\mu)]$  containing the origin. However, imposing both conditions  $\pi_0 = 0$  and  $\pi_1 = \pm 1$  does not work for nonlinear maps  $\Pi$ .



**Proposition 3.15.** *If  $\pi_0 = 0$ ,  $\pi_1 = \pm 1$  and  $v$  is a nontrivial solution then  $\pi_k = 0$  for all  $k \geq 2$ .*

*Proof.* Since  $v$  is nontrivial, there exists some  $j$  such that  $v_k = 0$  for all  $k < j$  and  $v_j \neq 0$ . Consider the entry  $m_{(j+1)j}$  and observe that it can be explicitly calculated

$$m_{(j+1)j} = (j+1)\pi_2.$$

Since  $v_k = 0$  for all  $k < j$  we must solve the following equation coming from the  $(j+1)$ -row of  $(M - \text{Id})v = 0$

$$(j+1)\pi_2 v_j + v_{j+1}(\pi_1^{j+2} - 1) = 0.$$

Since  $\pi_1 = 1$  or  $\pi_1 = -1$  it follows that  $\pi_2 = 0$  as  $v_j \neq 0$ . By induction on the minor diagonals of the matrix we see that  $\pi_2 = 0$  then implies  $\pi_3 = 0$  and so on.  $\square$

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